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ON THE REALIZATION OF CONSTRAINTS IN NONHOLONOMIC MECHANICS*

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The possibility of realizing nonholonomic constraints using large nonconservative forces is considered. Mechanical characteristics of some geometric objects investigated in /1/ are presented. This makes it possible to consider in a natural way the transition from the principle for systems without constraints to that of systems with constraints. Basic formulations are given in invariant form. An example is presented.

1. Consider a smooth dynamic system defined by the Lagrangian L, a smooth function on the tangential stratification TM of the configuration space M, which is equivalent to specifying the Hamiltonian H, a smooth function on the cotangent stratification T^*M. The Legendre representation $Z: TM \to T^*M$ corresponds to the Lagrangian L.

We denote the local coordinates in M by $q^1, \ldots, q^n$, in TM by $q^1, \ldots, q^n, q^{1\cdot}, \ldots, q^{n\cdot}$, and in T^*M by $q^1, \ldots, q^n, p^1, \ldots, p^n$; $p^i = \partial L/\partial q^{i}$. The nonconservative forces are defined in TM by the horizontal form ω, or by form $\omega^* = (Z^{-1})^*\omega$ in T^*M. In coordinate notation

$$\omega = \sum_{i=1}^{n} Q_i(q, q^{\cdot})\, dq^i, \quad \omega^* = \sum_{i=1}^{n} Q_i(q, p)\, dq^i$$

The system trajectories are integral curves of vector field X in T^*M, which is defined by the equation /2,3/

$$X \lrcorner\, \Omega = -dH + \omega^* \tag{1.1}$$

where Ω is a simplectic form in T^*M.

In coordinate form Eq.(1.1) is equivalent to Hamilton's equations with nonconservative forces

$$dq^i/dt = \partial H/\partial p^i, \quad dp^i/dt = -\partial H/\partial q^i + Q_i$$

If Lagrangian L is nondegenerate, Z is a local diffeomorphism. Then, if C is the integral curve of field X in T^*M and $C^* = (Z^{-1})^* C$ is a curve on TM, C^* is the integral curve of field $Y = (Z^{-1})_* X$ in TM. Consequently, the system trajectories are integral curves of field $Y = (Z^{-1})_* X$ in TM. Applying to formula (1.1) mapping Z^*, we obtain for field Y an equation of the form

$$Y \lrcorner\, \Omega_L + dH_L = \omega \tag{1.2}$$

where $\Omega_L = Z^*\Omega$ is the fundamental 2-nd form of the Lagrangian L and $H_L = Z^*H$ is the energy that corresponds to that Lagrangian. Equation (1.2) corresponds to the principle of d'Alambert /1/. In coordinate form this equation is a Legendre equation of the second kind

$$\frac{d}{dt}\frac{\partial L}{\partial q^{i\cdot}} - \frac{\partial L}{\partial q^i} = Q_i$$

If the Lagrangian L is nondegenerate and $I: \Lambda^1(T^*M) \to \mathrm{Vect}\,(T^*M)$ is a simplectic isomorphism, $I_L = (Z^{-1})_* \circ I \circ (Z^{-1})^*$, $I_L: \Lambda^1(TM) \to \mathrm{Vect}\,(TM)$ is an isomorphism of 1-forms and of vector fields, and Eq.(1.2) assumes the form $I_L^{-1}(Y) = -dH_L + \omega$, hence

$$Y = -I_L(dH_L) + I_L(\omega) \tag{1.3}$$

where $\Lambda^1(K)$ and $\mathrm{Vect}\,(K)$ are moduli of linear differential forms and of vector fields, respectively, in the manifold K (in our case K is TM and T^*M), and $I_L(\omega)$ is the vector field of force ω relative to the given mechanical system. In coordinate form (1.3) are Legendre equations that are solvable for derivatives.

If L is a Lagrangian of the mechanical type, i.e. $L = {}^1/_2 g_{ij} q^{i\cdot} q^{j\cdot} + U(q)$, where $G = {}^1/_2 g_{ij} dq^i \otimes dq^j$ is the Riemannian metric in M, and $U(q)$ is the force function, Eqs.(1.3) assume the form

$$dq^i/dt = q^{i\cdot}, \quad dq^{i\cdot}/dt = \Gamma^i_{kl}(q)\, q^{k\cdot} q^{l\cdot} + g^{is}(\partial U/\partial q^s + Q_s)$$

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*Prikl.Matem.Mekhan.,45,No.3,481-487,1981 where Γ_{kl}^{i} are the Christoffel symbols of Riemannian connectedness associated with metric G.

2. Let m independent linear nonholonomic constraints

$$h_j(q, q^{\cdot}) = \sum_{i=1}^{n} a_i^{n-m+j}, \quad q^{i\cdot} = 0, \quad j = 1, \ldots, m \tag{2.1}$$

be imposed on the system.

We assume the constraint to be defined by the m-dimensional codistribution D on M stretched over forms $\chi_j \in \Lambda^1(M)$ defined by the equalities $\chi_j(X)(a) = S_X^*(h_j)(a)$, where $S_X : M \to TM$ is the graph of an arbitrary cross section of X. In coordinate form

$$\chi_j = \sum_{i=1}^{n} a_i^{n-m+j} dq^i$$

The specification of distribution D is equivalent to specifying a $(n-m)$-dimensional distributions in M: in each tangent subspace T_aM is fixed a $(n-m)$-dimensional subspace $D_a(M)$, in which must lie the velocity vector.

It was shown in /4,5/ that a holonomic constraint may be defined as the limit case of a system with large potential energy. A particular case of realization of a nonholonomic constraint (the motion of Chaplygin's sled by inertia was considered in /6/. We shall consider the general case of linear nonholonomic constraints.

Let us substitute force

$$F = -\mu \sum_{j=1}^{m} h_j \pi^* \chi_j \tag{2.2}$$

for the nonholonomic constraint (2.1) which depends on parameter $\mu > 0$. In this equation π is the natural projection of TM on M. We also represent force (2.2) in the form $F = \tau d\Phi$ where the potential

$$\Phi = -\frac{1}{2}\mu \sum_{j=1}^{m} h_j^2$$

Operation $\tau : \Lambda^1(TM) \to \Lambda^1(TM)$ was defined in /1/ in coordinate notation $\tau : adq + bdq^{\cdot} \mapsto bdq$). Note that force F belongs to codistribution π^*D.

If L is a Lagrangian of the mechanical type, the vector field $F : I_LF$ of force F relative to the mechanical system considered is an acceptable geometric characteristic of that force. Since the form of F is horizontal, the field I_LF is vertical /1/. For any $\xi \in TM$ the isomorphism/7/ $I_\xi : T_aM \to T_\xi(T_aM)$, where $a = \pi(\xi)$, is determinate. In coordinate notation, when

$$Z_\xi = \sum_{i=1}^{n} A^i \frac{\partial}{\partial q^{i\cdot}}\Big|_\xi$$

then

$$I_\xi^{-1}(Z_\xi) = \sum_{i=1}^{n} A^i \frac{\partial}{\partial q^i}\Big|_{\pi(\xi)}$$

For any point $\xi \in TM$ vector $I_\xi^{-1}\cdot(I_LF)_\xi$ is orthogonal to the subspace $D_{\pi(\xi)}M$ in metric G. The equations of motion of the system with acting force F are of the form

$$X \rfloor \Omega = -dH + \omega^* + F^* \tag{2.3}$$

In coordinate notation

$$dq^i/dt = \partial H/\partial p^i$$

$$\frac{dp^i}{dt} = -\frac{\partial H}{\partial q^i} + Q_i(q, p) - \mu \sum_{j=1}^{m} h_j(q, p)\, a_i^{n-m+j}$$

We select the quasi-velocities $\pi^{1\cdot}, \ldots, \pi^{n\cdot}; \pi^{i\cdot} = a_k^i q^{k\cdot}$ so that $\pi^{n-m+j\cdot} = h_j$, $j = 1, \ldots, m$; $q^1, \ldots, q^n$, $\pi^{1\cdot}, \ldots (\pi^{n-m\cdot})$ is a system of coordinates of the submanifold $S = \{(q, q^{\cdot}) \in TM \mid h_j(q, q^{\cdot}) = 0\}$ and pass to coordinates $v^i = \partial L^*/\partial \pi^{i\cdot} = b_i^k p^k$ in T^*M, where $L^*(q, \pi^{\cdot}) = L(q, q^{\cdot}(q, \pi^{\cdot}))$, $a_l^i b_k^l = \delta_k^i$. In the system of coordinates $q^1, \ldots, q^n, v^1, \ldots, v^n$ Eqs. (2.3) are of the form

$$\frac{dq^i}{dt} = b_k{}^i \frac{\partial H^*}{\partial v^k}$$

$$\frac{dv^i}{dt} = -b_k{}^i \frac{\partial H^*}{\partial q^k} - \gamma_{ijk} v^j \frac{\partial H^*}{\partial v^k} + Q_k b_k{}^i - \mu \sum_{s=1}^{m} h_s \delta_i^{n-m+s}$$

The form of Eqs. (2.3) in the system of coordinates q, v implies that when $\mu = \infty$ they become the equations of motion of a system with constraints (2.1).

3. Using the notation $(x^1, \ldots, x^{2n}) = (q^1, \ldots, q^n, v^1, \ldots, v^n)$, $2n - m = l$ we write Eqs. (2.3) in the form

$$x^{i\cdot} = g_i(x^1, \ldots, x^{2n}), \quad i = 1, \ldots, l$$
$$\varepsilon (x^{l+j})^{\cdot} = \varepsilon g_{l+j}(x) + h_j(x), \quad j = 1, \ldots, m \tag{3.1}$$

where $\varepsilon = 1/\mu$ is a small parameter.

Further analysis is effected locally, assuming that system (3.1) is in region U of space R^{2n} of variables $x^1, \ldots, x^{2n}$. Unless otherwise stated, solutions will be considered with the initial condition $P_0 = (x_0{}^1, \ldots, x_0{}^{2n})$ on surface $\Gamma = \{x \mid h(x) = 0\}$.

Beside system (3.1) we consider the system

$$x^{i\cdot} = g_i(x)$$
$$\varepsilon_2 (x^{l+j})^{\cdot} = \varepsilon_1 g_{l+j}(x) + h_j(x) \tag{3.2}$$

whose solution we denote as follows:

$$x = \varphi(t, \varepsilon_1, \varepsilon_2) \tag{3.3}$$

Function $\varphi(t, \varepsilon, \varepsilon)$ is also a solution of system (3.1) (with the same initial condition P_0). Suppose that function g_i, h is analytic in region U. Then with a small ε_1 we can represent solution (3.3) in the form of series /8/

$$\varphi(t, \varepsilon_1, \varepsilon_2) = \varphi_0(t, \varepsilon_2) + \sum_{i=1}^{\infty} \varepsilon_1{}^i \varphi_i(t, \varepsilon_2) \tag{3.4}$$

where $\varphi_0(t, \varepsilon_2)$ is the solution of system (received from (3.2) when $\varepsilon_1 = 0$)

$$x^{i\cdot} = g_i(x), \quad \varepsilon_2 (x^{l+j})^{\cdot} = h_j(x)$$

Solution (3.3) was derived for $0 \leqslant \varepsilon_1 \leqslant \varepsilon_1{}^\circ$, $0 < \varepsilon_2 \leqslant \varepsilon_2{}^\circ$, and series (3.4) uniformly converges with respect to t for $0 \leqslant t \leqslant T$. In what follows various constants whose values are unimportant are denoted in like manner. We shall also consider the system

$$x^{i\cdot} = g_i(x), \quad \varepsilon_1 g_{l+j}(x) + h_j(x) = 0 \tag{3.5}$$

which is obtained from (3.2) for $\varepsilon_2 = 0$, and, also, the equation of rapid motions of system (3.2)

$$\varepsilon_2 (x^{l+j})^{\cdot} = \varepsilon_1 g_{l+j}(x) + h_j(x) \tag{3.6}$$

When L is a Lagrangian of mechanical type

$$G + {}^1/_2 g_{ij}\, dq^i \otimes dq^j = {}^1/_2 c_{ij}\, d\pi^i \otimes d\pi^j \tag{3.7}$$

which implies that

$$\partial h_i / \partial x^{l+j} = -\mu d_{n-m+i,\, n-m+j}, \quad i, j = 1, \ldots, m$$

where $\| d_{ij} \|$ is a matrix inverse of matrix $\| c_{ij} \|$. Since matrix $\| d_{ij} \|$ is positive definite, matrix $\| \partial h_i / \partial x^{l+j} \|$ is negative definite. Hence, if ε_1 is small, all roots of the characteristic equation of system (3.6) have negative real parts. Note that a similar proof also applies when L is an arbitrary positive definite Lagrangian.

Thus any point on surface $\varepsilon_1 g_{l+j}(x) + h_j(x) = 0$, $j = 1, \ldots, m$ represents an asymptotically stable equilibrium position of Eq. (3.6). Consequently the conditions of Tikhonov's theorem /8,9/ are satisfied, and for $0 < t \leqslant T$ there exists the limit $\lim_{\varepsilon_2 \to 0} \varphi(t, \varepsilon_1, \varepsilon_2) = \psi(t, \varepsilon_1)$ which uniformly converges with respect to t on any segment $[t_0, T]$, $0 < t_0 < T$, where $\psi(t, \varepsilon_1)$ is the solution of system (3.5) with the initial point $P_1 = (x_0{}^1, \ldots, x_0{}^l, x_1{}^{l+1}, \ldots, x_1{}^{2n})$ lying on surface $\varepsilon_1 g_{l+j}(x) + h_j(x) = 0$ (it can be assumed that $\psi(t, \varepsilon_1)$ is discontinuous with respect to x^{l+j} with initial condition P_0 /8/). As $\varepsilon_1 \to 0$ $P_1 \to P_0$ and there exists the limit $\lim_{\varepsilon_1 \to 0} \psi(t, \varepsilon_1) = \varphi_0(t)$, where $\varphi_0(t)$ is the solution of system $x^{i\cdot} = g_i(x)$, $h_j(x) = 0$; $i = 1, \ldots, l$; $j = 1, \ldots, m$.

By virtue of Tikhonov's theorem there exists the limit $\lim_{\varepsilon_2\to 0}\varphi_0(t,\varepsilon_2)=\varphi_0(t)$ uniform with respect to t for $0\leqslant t\leqslant T$.

Series (3.4) is in powers of ε_1 and converges when $\varepsilon_1=\varepsilon_1^\circ$, hence its radius of convergence $r\geqslant\varepsilon_1^\circ$ (we assume that $r>\varepsilon_1^\circ$ and that series (3.4) converge absolutely when $\varepsilon_1<r$). According to the Cauchy—Hadamard formula

$$\overline{\lim_{n\to\infty}}\sqrt[n]{|\varphi_n(t,\varepsilon_2)|}=\frac{1}{r}<\frac{1}{\varepsilon_1^\circ}=c$$

Consequently, beyond some ordinal number $N\,|\varphi_n(t,\varepsilon_2)|<c^n$. If $\varepsilon_1<1/(2c)$, then for any $M>0$

$$\overline{\lim_{\varepsilon_2\to 0}}\sum_{j=N+1}^{N+M}\varepsilon_1{}^j\varphi_j(t,\varepsilon_2)\leqslant\sum_{j=N+1}^{N+M}\left(\frac{1}{2}\right)^j<k$$

Hence there exists the finite limit

$$\overline{\lim_{\varepsilon_2\to 0}}\sum_{j=N+1}^{\infty}\varepsilon_1{}^j\varphi_j(t,\varepsilon_2)$$

Similarly there exists the limit

$$\underline{\lim_{\varepsilon_2\to 0}}\sum_{j=N+1}^{\infty}\varepsilon_1{}^j\varphi_j(t,\varepsilon_2)$$

Also for $0\leqslant\varepsilon_1\leqslant\varepsilon_1^\circ, 0<t\leqslant T$ we have the limit

$$\overline{\lim_{\varepsilon_2\to 0}}\varphi(t,\varepsilon_1,\varepsilon_2)=\underline{\lim_{\varepsilon_2\to 0}}\varphi(t,\varepsilon_1,\varepsilon_2)=\lim_{\varepsilon_2\to 0}\varphi(t,\varepsilon_1,\varepsilon_2)$$

but in that case there exist the finite limits

$$\overline{\lim_{\varepsilon_2\to 0}}\sum_{j=0}^{N}\varepsilon_1{}^j\varphi_j(t,\varepsilon_2),\quad \underline{\lim_{\varepsilon_2\to 0}}\sum_{j=0}^{N}\varepsilon_1{}^j\varphi_j(t,\varepsilon_2)$$

i.e. there exist in some interval $(0,\varepsilon_2)$

$$\left|\sum_{j=0}^{N}\varepsilon_1{}^j\varphi_j(t,\varepsilon_2)\right|\leqslant c$$

Choosing an arbitrary set from N different numbers ε_1, $0<\varepsilon_1<\varepsilon_1^\circ$, we obtain a system linear with respect to functions $\varphi_j(t,\varepsilon_2)$, with a nonzero determinant. The boundedness of functions $\varphi_j(t,\varepsilon_2), j=1,\ldots,N$ when $0<\varepsilon_2\leqslant\varepsilon_2^\circ$, $0<t\leqslant T$ is proved by solving that system. Let us now consider series (3.4) when $\varepsilon_1=\varepsilon_2=\varepsilon$. When $0<\varepsilon\leqslant\varepsilon^\circ, 0<t\leqslant T$ it is majorated by the convergent series

$$\sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^i$$

Hence series (3.4) is uniformly convergent with respect to ε. When $0<\varepsilon\leqslant\varepsilon^\circ$, $0<t\leqslant T$ we have the limit $\lim_{\varepsilon\to 0}\varepsilon^j\varphi_j(t,\varepsilon)=0$. It is then possible to pass to limit term-by-term, and consequently, for $0<t\leqslant T$ there exists the limit

$$\lim_{\varepsilon\to 0}\varphi(t,\varepsilon,\varepsilon)=\varphi_0(t) \tag{3.8}$$

When $t=0$, $\varphi(0,\varepsilon,\varepsilon)=\varphi_0(t)$ so that equality (3.8) is satisfied for $0\leqslant t\leqslant T$. The estimate of $|\varphi(t,\varepsilon,\varepsilon)-\varphi_0(t)|$ on segment $[0,T]$ shows that the convergence is uniform.

Proof of the theorem in Sect.3 is simplified by using the results of /10/.

4. The preceding considerations imply that force $F(\mu)$ realizes the nonholonomic constraint (2.1). This means that when $q^i=q^i(t,\mu)$ is the trajectory of system (2.3) with initial condition P_0 determined on segment $0\leqslant t\leqslant T$, there exists the uniform with respect to t limit

$$\lim_{\mu\to\infty}q(t,\mu)=q(t)$$

The limit function $q(t)$ is the trajectory of a system with constraint (2.1), i.e. at large values of parameter μ the trajectory of the system with acting force $F(\mu)$ and the system with the nonholonomic constraint (2.1) are close. At transition to limit as $(\mu\to\infty)$ the trajectories are the same, with the mean value of force F oscillating about S is the reaction force R of the nonholonomic constraint. Force R belongs to the codistribution π^*D. Thus naturally arises the codistribution in which lie the nonholonomic constraint reaction forces. For Lagrangians of the mechanical type the geometric characteristic of 1-forms belonging to the codistribution π^*D was given in Sect.2. The constraint realized by force $F(\mu)$ is ideal. Indeed, the virtual displacement is determined as the vector field $T(Z)$ in TMsuch that field Z is cancelled by the codistribution D. But the codistribution π^*D cancels field $T(Z)$,which means that the work of the constraint reaction force over the virtual displacement is zero.

Example. For small plate with knife edge on an inclined plane /6/ the equation of nonholonomic constraint is of the form

$$v = -x^{\cdot}\sin\varphi + y^{\cdot}\cos\varphi = 0 \tag{4.1}$$

We substitute a force dependent on parameter μ for constraint (4.1). The equations of motions in quasi-coordinates are of the form

$$\begin{aligned} &u^{\cdot} = v\omega + g\sin\alpha\cos\varphi \\ &v^{\cdot} = -u\omega - g\sin\alpha\sin\varphi - \mu v, \quad \omega^{\cdot} = 0 \\ &u = x^{\cdot}\cos\varphi + y^{\cdot}\sin\varphi \\ &v = -x^{\cdot}\sin\varphi + y^{\cdot}\cos\varphi, \quad \omega = \varphi^{\cdot} \end{aligned} \tag{4.2}$$

Solving system (4.2) with initial conditions $x(0) = y(0) = \varphi(0) = x(0)^{\cdot} = y^{\cdot}(0) = 0,\ \varphi^{\cdot}(0) = \omega_0$ and passing to the limit $(\mu \to \infty)$, we obtain

$$x = \frac{g\sin\alpha}{2\omega_0^2}\sin^2\omega_0 t, \qquad y = \frac{g\sin\alpha}{2\omega_0^2}\left(\omega_0 t - \frac{1}{2}\sin 2\omega_0 t\right), \qquad \varphi = \omega_0 t \tag{4.3}$$

Equations (4.3) are the equations of motion of the system with the nonholonomic constraint (4.1).

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